$\boxed{①(a)}$

$f(x) = x^3 + x \rightarrow f(1) = 2$

$f'(x) = 3x^2 + 1 \rightarrow f'(1) = 4$

$f''(x) = 6x \rightarrow f''(1) = 6$

$f'''(x) = 6 \rightarrow f'''(1) = 6$

$f^{(k)}(x) = 0 \rightarrow f^{(k)}(1) = 0$

$k \geq 4$ $\qquad$ $k \geq 4$

So, $x^3 + x = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$

$= 2 + 4(x-1) + 3(x-1)^2 + 1 \cdot (x-1)^3$

with radius of convergence $r = \infty$

Thus it converges for $-\infty < x < \infty$

$\boxed{①(b)}$ $f(x) = x$

You can use the same method as above or just notice that

$$\underbrace{x}_{f(x)} = 1 + (x-1)$$

This has radius of convergence $r = \infty$ and converges for $-\infty < x < \infty$

$\boxed{①(c)}$

$f(x) = x^2 \longrightarrow f(1) = 1$

$f'(x) = 2x \longrightarrow f'(1) = 2$

$f''(x) = 2 \longrightarrow f''(1) = 2$

$f^{(k)}(x) = 0$ for $k \geq 3$ $\longrightarrow f^{(k)}(1) = 0$ for $k \geq 3$

So,

$$x^2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$= 1 + 2(x-1) + (x-1)^2$$

This has radius of convergence $r = \infty$
and converges for $-\infty < x < \infty$

1(d)

$$f(x) = \frac{-1}{1-x^2} = -\left(1 + x^2 + (x^2)^2 + (x^2)^3 + \dots\right)$$

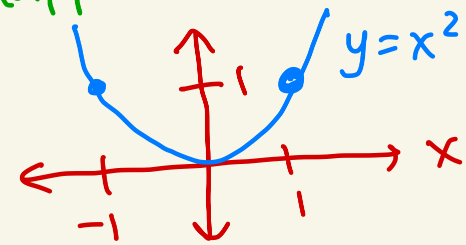
Geometric sum:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

$-1 < u < 1$ or $|u| < 1$

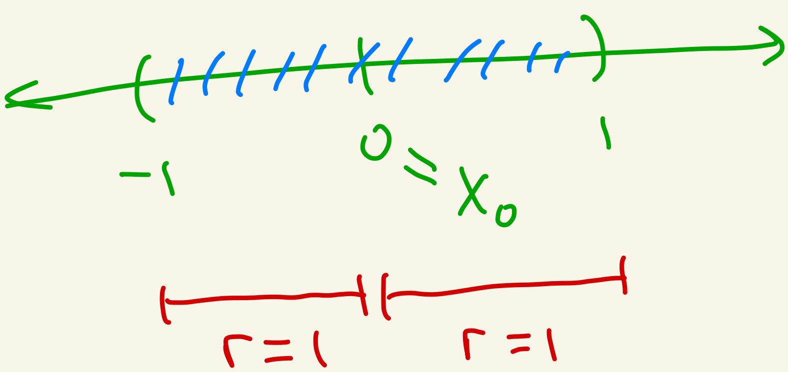
need $-1 < x^2 < 1$ to use the geometric sum formula

This happens when $-1 < x < 1$



$$= -1 - x^2 - x^4 - x^6 - \dots$$

has radius of convegence $r = 1$ since it converges when $-1 < x < 1$.



1(e)

$$f(x) = \frac{x}{1-x^2} = x\left(\frac{1}{1-x^2}\right)$$

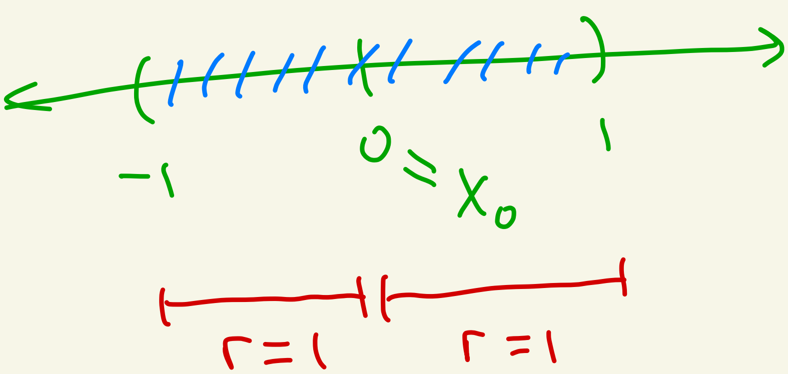
$$= x\left(1 + x^2 + (x^2)^2 + (x^2)^3 + \dots\right)$$

$$= x + x^3 + x^5 + x^7 + \dots$$

from previous problem

when $-1<x<1$

has radius of convegence $r=1$ since it converges when $-1 < x < 1$.



$\boxed{1(f)}$ Find a power series expansion for $f(x) = \frac{1}{x}$ at $x_0 = 1$.

If we only look at $x > 0$, then

$$\frac{1}{x} = \frac{d}{dx}\ln(x)$$

$$= \frac{d}{dx}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n\right]$$

$\boxed{0<x<2}$

$$= \frac{d}{dx}\left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots\right]$$

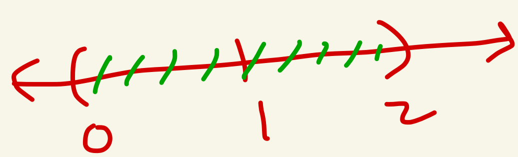
$$= 1 - (x-1) + (x-1)^2 - \cdots$$

So,

$$\frac{1}{x} = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^{n-1} = 1 - (x-1) + (x-1)^2 - \cdots$$

which as radius of convergence $r = 1$ about $x_0 = 1$.

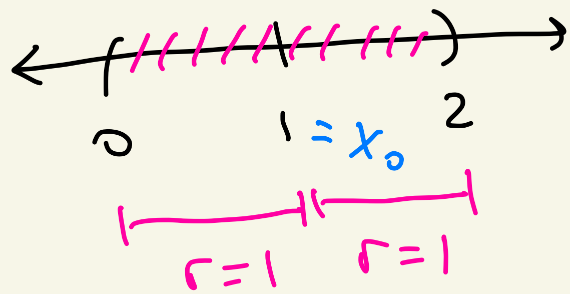
So the series converges for $0 < x < 2$



① (g) From the previous problem we know that

$$\frac{1}{x} = \sum_{n=1}^{\infty} (-1)^{n+1}(x-1)^{n-1} = 1-(x-1)+(x-1)^2-(x-1)^3+\dots$$

when $0<x<2$, ie radius of convergence $r=1$.



Differentiating both sides

$$-\frac{1}{x^2} = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-1)^{n-2}$$

$$= -1+2(x-1)-3(x-1)^2+4(x-1)^3-\dots$$

when $0<x<2$.

Thus,

$$\frac{1}{x^2} = \sum_{n=2}^{\infty} (-1)^{n+2}(n-1)(x-1)^{n-2}$$

$$= 1-2(x-1)+3(x-1)^2-4(x-1)^3+\dots$$

for $0<x<2$ with radius of convergence $r=1$.

1(h)

We know that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

for $-\infty < x < \infty$, ie radius of convergence $r = \infty$.

Plug x^2 into the formula to get:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^2)^n = 1 + x^2 + \frac{1}{2!}(x^2)^2 + \frac{1}{3!}(x^2)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} = 1 + x^2 + \frac{1}{2!} x^4 + \frac{1}{3!} x^6 + \dots$$

for $-\infty < x < \infty$, ie radius of convergence $r = \infty$.

$\boxed{\text{② (a)}}$ From class,

$$\sin(x) = \underbrace{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + }_{\text{first 5 terms}}\frac{1}{9!}x^9 - \ldots$$

for $-\infty < x < \infty$.

Thus an estimate for $\sin(0.1)$ is

$$0.1 - \frac{1}{3!}(0.1)^3 + \frac{1}{5!}(0.1)^5 - \frac{1}{7!}(0.1)^7$$

$$= 0.1 - \frac{1}{6}(0.001) + \frac{1}{120}(0.00001) - \frac{1}{5040}(0.0000001)$$

$$= \boxed{0.0998334}$$

$\boxed{\text{(b)}}$ My calculator says that

$$\sin(0.1) \approx \boxed{0.0998334166468 28\ldots}$$

We were very close!

3 (a) From class,

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

$$= \underbrace{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}_{\text{first 4 terms}} + \frac{1}{5}(x-1)^5 - \ldots$$

This is valid for $0 < x < 2$.

Thus an estimate for $\ln(1.1)$ is

$$(1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4$$

$$= 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4}$$

$$= \boxed{0.0953083}$$

(b) My calculator says that

$$\boxed{\ln(1.1) \approx 0.09531018\ldots}$$

That's pretty close, 4 decimal places of accuracy!